Global surface of section and Hamiltonian dynamics on 2-disk

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CRM - Université de Montréal

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Goal

The goal of this talk is to prove the following result.

Theorem (Abbondandolo-Bramham-Hryniewicz-Salomão, 2018)

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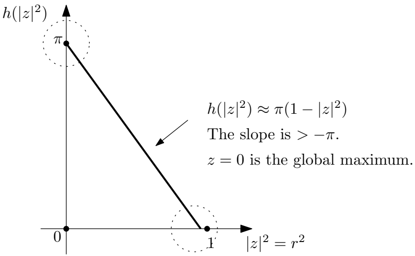
$$\mathrm{vol}_{\alpha\wedge d\alpha}(S^3) = \pi^2 + \mathrm{Cal}(\phi). \tag{1}$$

Moreover, this ϕ is special since $\mathrm{Cal}(\phi) < 0$ but $T_{\min}(\alpha) \geq \pi$.

Radial symmetric Hamiltonian on $\mathbb{D}$

- Consider the autonomous Hamiltonian $H : \mathbb{D} \to \mathbb{R}$ such that $H(z) = h(|z|^2)$ where $h : [0, 1] \to \mathbb{R}$ is defined in the following picture (where circle regions are smoothed),



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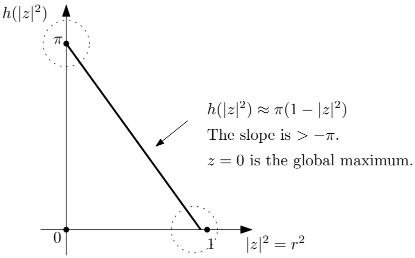

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(1) $\phi_H^1(z) = e^{-2h'(|z|^2)i} z$.

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In general,

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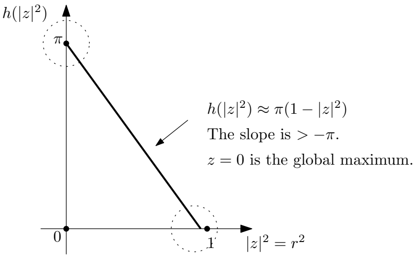

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- Observe that fixed points lie in neighborhood of $\partial\mathbb{D}$ and $z = 0$.

Action function

Given a radial symmetric Hamiltonian function $H : \mathbb{D} \to \mathbb{R}$, where $H(z) = h(|z|^2)$. Consider **action function** $\sigma : \mathbb{D} \to \mathbb{R}$ defined by

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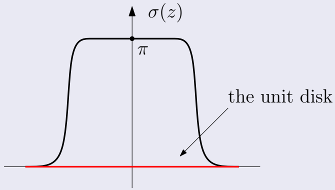
Example

For H chosen earlier, action function is the following graph.



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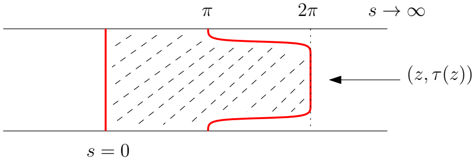

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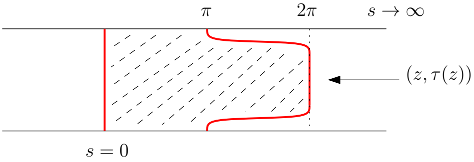

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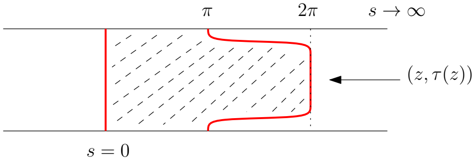

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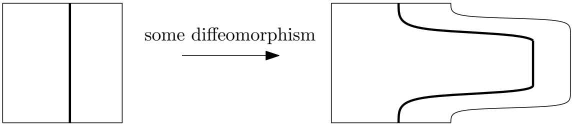
- Near the boundary, $M = U \times \mathbb{R}/\pi\mathbb{Z}$ where U is a neighborhood of $\partial\mathbb{D}$ and $\eta = \lambda + ds$.

Bramham's construction (II)

The desired contact form β is on the solid torus $\mathbb{D} \times \mathbb{R}/\pi\mathbb{Z}$. There exists some diffeomorphism $f : \mathbb{D} \times \mathbb{R}/\pi\mathbb{Z} \to M$ as in the following picture.



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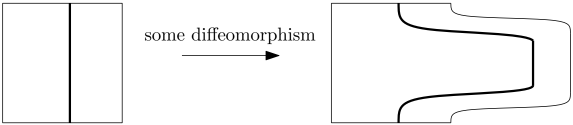

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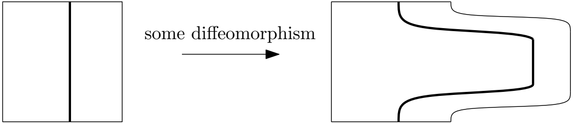

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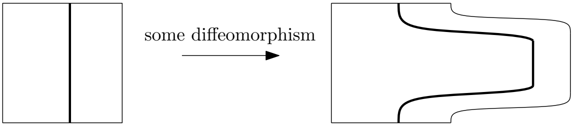

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Example

One can compute that $\mathrm{vol}_{\beta\wedge d\beta}(\mathbb{D} \times \mathbb{R}/\pi\mathbb{Z}) = \mathrm{vol}_{\eta\wedge d\eta} M \simeq 2\pi^2$*. Then, up to small gaps, we get*

$$\mathrm{vol}_{\beta\wedge d\beta}(\mathbb{D} \times \mathbb{R}/\pi\mathbb{Z}) = \pi^2 + \mathrm{Cal}(\phi_H^1).$$

Global surfaces of section in S^3

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- $\mathrm{vol}_{\alpha\wedge d\alpha}(S^3) = \mathrm{vol}_{\beta\wedge d\beta}(\mathbb{D} \times \mathbb{R}/\pi\mathbb{Z})$, which implies the volume-Calabi equation (1). Moreover, $T_{\min}(\alpha) = \pi$ (Exercise).

A cheap variation

- So far, we have seen an example that some $\phi \in \mathrm{Ham}_c(\mathring{\mathbb{D}}, \omega)$ results in a contact 1-form α on S^3 which can satisfy the volume-Calabi equation (1) and $T_{\min}(\alpha) = \pi$, but unfortunately $\mathrm{Cal}(\phi) > 0$.

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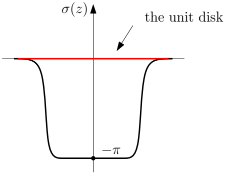

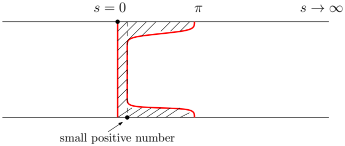

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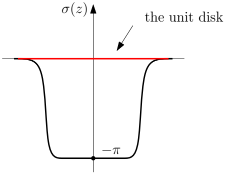

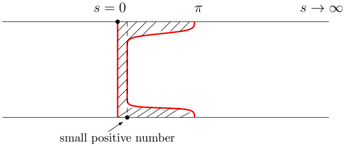

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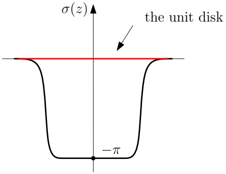

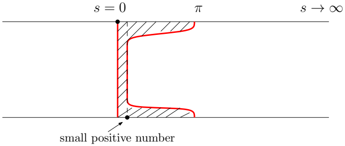

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• Need a new example to satisfy both $\mathrm{Cal}(\phi) < 0$ and $T_{\min}(\alpha) \geq \pi$.

Sinkhole (cf. Usher's Banach-Mazur distance paper)

Example

Consider a disk inside $\mathbb{D}$*, denoted by* $\mathbb{D}_{(z_0,r_0)} = \{|z - z_0| \leq r_0\}$

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An estimation: $|u \circ (\psi^{-1} \circ \phi) - u \circ \psi^{-1}| \leq r_0 |z_0| \leq r_0$, which implies that

$$-k - r_0 \leq \sigma_\phi \leq r_0. \tag{2}$$

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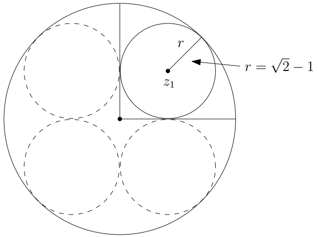
Consider sinkholes $\mathbb{D}_i$ for $i = 1, \ldots, 4$, sitting inside $\mathbb{D}$ in the following symmetric way.



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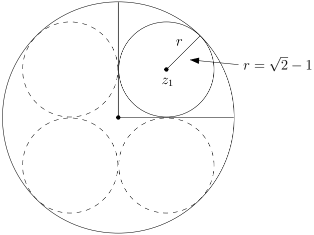

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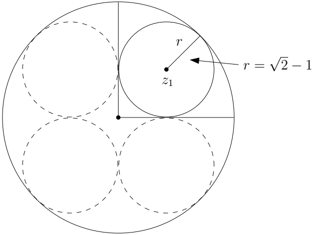

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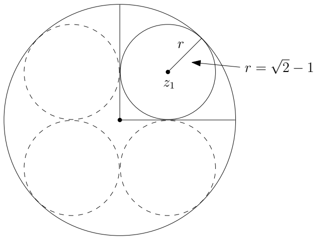

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Then we can compute

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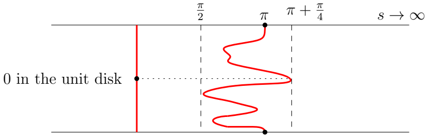
A schematic picture of Bramham's solid torus (denoted earlier by M) is the following.



Final step: estimation of $T_{\min}(\alpha)$

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The upshot is:

$$\begin{cases} \mathrm{vol}_{\alpha\wedge d\alpha}(S^3) = \pi^2 + \mathrm{Cal}(\phi) \\ \mathrm{Cal}(\phi) < 0 \\ T_{\min}(\alpha) \geq \pi \end{cases} \Rightarrow T_{\min}(\alpha)^2 > \mathrm{vol}_{\alpha\wedge d\alpha}(S^3).$$

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• One can re-do the computation above by starting with a very narrow sector of $\mathbb{D}$ and rotate it making it symmetric. Then the resulting ϕ will be C^0-close to $\mathbb{1}_{\mathbb{D}}$ (and α is C^0-closed to α_0).

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Its proof runs our construction backwards, i.e., constructing an ω-preserving diffeomorphism on $\mathbb{D}$ from a contact form on S^3.

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• Usher's tube construction: given a Liouville domain (W, λ) and an autonomous Hamiltonian function $H : W \to \mathbb{R}$ satisfying certain "positivity condition", the following construction results in another Liouville domain,

$$W_H := \{(w, z) \in W \times \mathbb{C} \,|\, \pi |z|^2 \leq H(w)\}.$$